Structural Compression and Hidden Generators

A Conditional Information Invariant for Finite Mathematical Universes

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In memoriam François Englert†

Abstract

We introduce a finite information-theoretic invariant measuring the structural complexity of mathematical objects relative to a chosen family of constraints. Recognizing that mathematical constraints form dense dependency networks, we utilize permutation-averaged conditional self-information to construct an order-independent coherence metric. We prove basic stability and logarithmic representation results for finite constraint systems. Furthermore, we connect this invariant to algorithmic information theory by defining the Structural Compression Ratio, framing highly structured mathematical objects as compression points in a constraint space. The framework formalizes a quantitative principle aligned with the Langlands program: mathematical objects discovered to be massive compression points of apparent independent truths frequently admit simpler generating descriptions in deeper geometric or representation-theoretic categories.

1 Motivation

The historical development of mathematics frequently reveals a pattern: complex mathematical objects that resist analysis within their apparent domains often yield entirely to the discovery of a new conceptual language that natively absorbs their defining constraints.

The purpose of this paper is to construct a decent meta-mathematical invariant that formalizes this structural completeness. We introduce an information-theoretic invariant that measures an object's structural complexity via conditional probability. By formulating this functional over finite universes and bridging it with algorithmic information theory, we provide a quantitative ranking invariant that shifts the evaluation of mathematical completeness from heuristic observation to formal metric analysis.

2 Finite Constraint Networks and Conditional Information

To ensure the invariant is well-defined without relying on arbitrary topological assumptions over infinite classes of functions, we restrict our analysis to finite comparison sets.

Definition 2.1 (Finite Coherence Universe)**.** Let

$$\mathcal{U}_N = \{L_1, \ldots, L_N\} \tag{1}$$

be any finite collection of mathematical objects under comparison. A probability measure

$$\mu : \mathcal{U}_N \to [0, 1] \tag{2}$$

is any probability distribution satisfying $\sum_{L \in \mathcal{U}_N} \mu(L) = 1$. All subsequent definitions are strictly relative to the chosen pair $(\mathcal{U}_N, \mu)$.

Definition 2.2 (Conditional Information Network)**.** A computable boolean constraint on the universe is a predicate $C : \mathcal{U}_N \to \{0, 1\}$. Modern mathematics is characterized by implication networks; constraints are rarely logically independent.

Let π be a permutation of the indices $\{1, \ldots, n\}$, defining an ordered sequence of constraints $\mathcal{C}_\pi = (C_{\pi(1)}, \ldots, C_{\pi(n)})$. The conditional probability of constraint $C_{\pi(i)}$ given the satisfaction of all preceding constraints in the ordering π is:

$$p_{\pi(i)} = \mu(C_{\pi(i)} \mid C_{\pi(1)}, \ldots, C_{\pi(i-1)}) \tag{3}$$

This formulation measures exactly how much *new* restriction is imposed by $C_{\pi(i)}$ upon the subspace already carved out by the preceding network.

3 The Order-Independent Coherence Index

Because conditional information allocation depends strictly on the order in which constraints are evaluated, a naive summation over an arbitrary ordering fails to be a true invariant. We rectify this by importing permutation averaging.

Definition 3.1 (Ordered Coherence Score)**.** The ordered coherence score of an object $L \in \mathcal{U}_N$ with respect to the permutation π is:

$$\mathcal{S}_\pi(L) = \sum_{i=1}^{n} -\log_2 \left(p_{\pi(i)}\right) C_{\pi(i)}(L) \tag{4}$$

where the i-th term evaluates to 0 if any preceding constraint fails for L.

Definition 3.2 (Permutation-Averaged Coherence Invariant)**.** The *Structural Coherence Index* $\mathcal{S}_{\text{inv}}(L)$ is the expected value of the ordered score over all possible $n!$ permutations of the constraint system:

$$\mathcal{S}_{\text{inv}}(L) = \frac{1}{n!} \sum_{\pi \in S_n} \mathcal{S}_\pi(L) \tag{5}$$

Theorem 3.3 (Permutation Averaging Invariance)**.** *The averaged conditional coherence score $\mathcal{S}_{inv}(L)$ is strictly independent of any initial constraint ordering.*

Proof. Let $\sigma \in S_n$ be any arbitrary permutation of the constraint labels. Then the reordered invariant is:

$$\mathcal{S}'_{\text{inv}} = \frac{1}{n!} \sum_{\pi \in S_n} \mathcal{S}_{\sigma \circ \pi} \tag{6}$$

Since the map $\pi \mapsto \sigma \circ \pi$ acts as a bijection on the symmetric group S_n, the set of permutations $\{\sigma \circ \pi : \pi \in S_n\}$ is exactly S_n. Therefore:

$$\mathcal{S}'_{\text{inv}} = \frac{1}{n!} \sum_{\pi \in S_n} \mathcal{S}_{\pi} = \mathcal{S}_{\text{inv}} \tag{7}$$

This renders $\mathcal{S}_{\text{inv}}(L)$ a permutation-invariant functional of the constraint network. □

Theorem 3.4 (Logarithmic Characterization of Information Weights)**.** *Suppose $F : (0, 1] \to \mathbb{R}_{\geq 0}$ is a constraint-weighting functional on event probabilities satisfying:*

1. ***Continuity:*** *$F(p)$ is a continuous function.*
2. ***Multiplicativity for Independent Events:*** *For independent probabilities p and q, the functional is additive: $F(pq) = F(p) + F(q)$.*
3. ***Normalization:*** *For an event with probability $1/2$, $F(1/2) = k > 0$.*

Then $F(p)$ is uniquely given by the logarithmic measure $F(p) = -k \log_2 p$.

Proof. By the additivity condition, F satisfies Cauchy's functional equation $f(xy) = f(x) + f(y)$ for all $x, y \in (0, 1]$. It is a standard result in real analysis that the only continuous solutions to this functional equation are of the form $c \ln p$. The normalization condition fixes the constant c, yielding $F(p) = -k \log_2 p$. Thus, the chosen logarithmic weighting of the index is the unique continuous, additive measure of probabilistic information. □

4 Algorithmic Compression in Constraint Space

This metric directly connects the structural completeness of analytic objects to algorithmic information theory. A mathematical object can be viewed as a compression point in a space of constraints.

Let $K(L)$ denote the Kolmogorov complexity of L, defined informally as the length of the shortest program required to compute L to arbitrary precision.

Definition 4.1 (Structural Compression Ratio)**.** We define the Structural Compression Ratio $\Gamma(L)$ as the ratio of the conditional information explained by the object to its foundational description length:

$$\Gamma(L) = \frac{\mathcal{S}_{\text{inv}}(L)}{K(L)} \tag{8}$$

Remark 4.2 (On Encoding). In the present finite framework, $K(L)$ is interpreted relative to a fixed universal description language for computable analytic objects. Because Kolmogorov complexity is theoretically invariant only up to an additive constant $\mathcal{O}(1)$ dependent on the choice of the universal Turing machine, the resulting ratio $\Gamma(L)$ relies on a fixed canonical encoding to permit precise relative comparisons.

An algorithmically incompressible object requires vast amounts of independent, decoupled information to parameterize ($K(L)$ is large) yet satisfies few rigid analytic constraints ($\mathcal{S}_{\text{inv}}$ is small), yielding a low $\Gamma(L)$. By contrast, a highly structured object can be described compactly because a single hidden mechanism—such as a deeper geometric manifold—simultaneously generates multiple intersecting analytic constraints. Objects with high $\Gamma(L)$ scores are dense compression points of mathematical truth.

5 Computed Examples

To demonstrate that $\mathcal{S}_{\text{inv}}(L)$ functions as an active quantitative ranking invariant, we construct a finite universe $\mathcal{U}$ equipped with a uniform measure $\mu(L) = 1/4$:

$$\mathcal{U} = \{\zeta(s), E(z,s), \zeta(s,a), \mathrm{Li}(s)\} \tag{9}$$

where $\zeta(s)$ is the Riemann zeta function, $E(z,s)$ is the Epstein zeta function for a generic lattice, $\zeta(s,a)$ is the Hurwitz zeta function (for generic a), and $\mathrm{Li}(s)$ is the logarithmic integral.

We evaluate them against a constraint network $\mathcal{C} = \{C_1, C_2, C_3\}$:

- C_1: Admits meromorphic continuation to $\mathbb{C}$.
- C_2: Admits a completed functional equation arising from an automorphic or arithmetic symmetry.
- C_3: Admits a globally convergent Euler product.

The constraint satisfaction vectors (C_1, C_2, C_3) for the objects are:

- $\mathrm{Li}(s)$: $(0,0,0)$
- $\zeta(s,a)$: $(1,0,0)$
- $E(z,s)$: $(1,1,0)$
- $\zeta(s)$: $(1,1,1)$

To calculate $\mathcal{S}_{\text{inv}}(L)$, we evaluate the conditional self-information scores across all $3! = 6$ permutations. Note that for any permutation, the sequence halts and yields zero weight for all subsequent terms if a prior constraint fails.

Note that when a conditional weight evaluates to 0.000 (as seen in permutations where C_2 follows $C_1 \cap C_3$), it indicates that the constraint contributes zero novel information because it becomes certain given the preceding constraints ($\mu(C_2 \mid C_1 \cap C_3) = 1$).

Permutation π	**Cond. Weights** $(w_{\pi(1)}, w_{\pi(2)}, w_{\pi(3)})$	$\mathcal{S}_\pi(\mathbf{Li})$	$\mathcal{S}_\pi(\zeta(s,a))$	$\mathcal{S}_\pi(E)$	$\mathcal{S}_\pi(\zeta)$
$(1,2,3)$	$(0.415, 0.585, 1.000)$	0	0.415	1.000	2.000
$(1,3,2)$	$(0.415, 1.585, 0.000)$	0	0.415	0.415	2.000
$(2,1,3)$	$(1.000, 0.000, 1.000)$	0	0.000	1.000	2.000
$(2,3,1)$	$(1.000, 1.000, 0.000)$	0	0.000	1.000	2.000
$(3,1,2)$	$(2.000, 0.000, 0.000)$	0	0.000	0.000	2.000
$(3,2,1)$	$(2.000, 0.000, 0.000)$	0	0.000	0.000	2.000
Average $\mathcal{S}_{\mathbf{inv}}$		0.000	≈ 0.138	≈ 0.569	2.000

The invariant quantifies the dependency hierarchy. The transition from $E(z,s)$ to $\zeta(s)$ registers as an immense leap in coherence ($\approx 0.569 \to 2.000$) because the emergence of an Euler product, conditional upon permutations that first evaluate the functional equation, represents a massive injection of novel arithmetic compression. Conversely, objects like $\zeta(s,a)$ that satisfy early analytic constraints but fail deep structural symmetries are heavily penalized by the permutation averaging.

6 Hypothesis

We conclude by formalizing the core mathematical principle underlying this framework, generalizing the observations of the computed universe.

Conjecture (The Compression Principle). The Structural Compression Ratio $\Gamma(L)$ is maximized by mathematical objects that admit minimal generating categories.

When many constraints appear independent at the observational level but are generated by a smaller hidden structure, conditional information decreases relative to the number of observed consequences. Therefore, objects discovered to be immense compression points of apparent independent truths—those driving $\Gamma(L)$ to its supremum—frequently admit simpler generating descriptions in deeper mathematical frameworks (e.g., a geometric manifold, a cohomological theory, or a spectral operator).

This principle formalizes a fundamental mechanism of modern mathematics: just as elliptic curves and modular forms were unified, extreme structural coherence identifies the exact loci where disparate analytic properties collapse into a singular geometric origin.